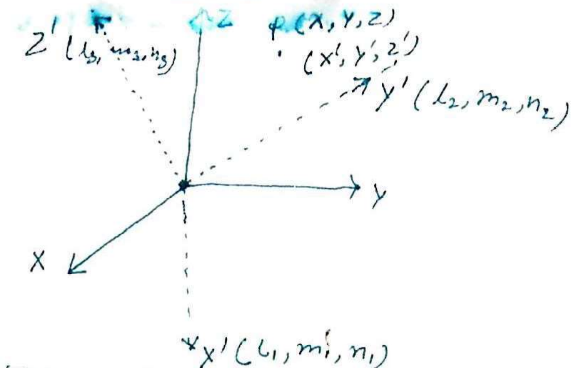
	x	y	z
x'	l_1	m_1	n_1
y'	l_2	m_2	n_2
z'	l_3	m_3	n_3



with reference to new system ox', oy', oz' of $oxyz$ having d-cs (l_1, m_1, n_1); (l_2, m_2, n_2); (l_3, m_3, n_3) of the dashed system the expression

$$\tau_{xx} x^2 + \tau_{yy} y^2 + \tau_{zz} z^2 = \tau_{xx}(l_1 x' + l_2 y' + l_3 z')^2 + \tau_{yy}(m_1 x' + m_2 y' + m_3 z')^2$$
$$+ \tau_{zz}(n_1 x' + n_2 y' + n_3 z')^2$$
$$= \tau_{x'x'} x'^2 + \tau_{y'y'} y'^2 + \tau_{z'z'} z'^2 + 2\tau_{y'z'} y'z'$$
$$+ 2\tau_{z'x'} z'x' + 2\tau_{x'y'} x'y' \quad \ldots (6)$$

similarly, $e_{xx} x^2 + e_{yy} y^2 + e_{zz} z^2 = e_{x'x'} x'^2 + e_{y'y'} y'^2 + e_{z'z'} z'^2$
$$+ 2 e_{y'z'} y'z' + 2 e_{z'x'} z'x' + 2 e_{x'y'} x'y' \quad \ldots (7)$$

since $x^2 + y^2 + z^2 = x'^2 + y'^2 + z'^2 \quad \ldots (8)$

and $\theta = e_{xx} + e_{yy} + e_{zz}$ being invariant i.e.

$$\theta = e_{xx} + e_{yy} + e_{zz} = e_{x'x'} + e_{y'y'} + e_{z'z'} = \theta' \ldots (9)$$

using (6), (7), (8) and (9) in (5), we get

$$\tau_{x'x'} x'^2 + \tau_{y'y'} y'^2 + \tau_{z'z'} z'^2 + 2\tau_{y'z'} y'z' + 2\tau_{z'x'} z'x' + 2\tau_{x'y'} x'y'$$
$$= \lambda \theta'(x'^2 + y'^2 + z'^2) + 2\mu\left[e_{x'x'} x'^2 + e_{y'y'} y'^2 + e_{z'z'} z'^2\right.$$
$$\left. + 2 e_{y'z'} y'z' + 2 e_{z'x'} z'x' + 2 e_{x'y'} x'y'\right] \ldots (10)$$

since (x', y', z') is any arbitrary point P so the co-efficients of x'^2, y'^2, z'^2, $x'y'$, $y'z'$, $z'x'$ on either side of eq^n (10) must be equal

so $\tau_{x'x'} = +\lambda\theta' + 2\mu e_{x'x'}$; $\tau_{y'y'} = +\lambda \cdot \theta' + 2\mu e_{y'y'}$

$\tau_{z'z'} = +\lambda\theta' + 2\mu e_{z'z'}$; $\tau_{y'z'} = 2\mu e_{y'z'}$

$\tau_{z'x'} = 2\mu e_{z'x'}$; $\tau_{x'y'} = 2\mu e_{x'y'}$

which are the stress-strain relation for an isotropic elastic medium. Replacing x', y', z' by 1, 2, 3 these relations can be put in the form

$$\tau_{ij} = \lambda \theta \delta_{ij} + 2\mu e_{ij}$$

where $\theta = e_{11} + e_{22} + e_{33}$

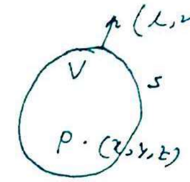
$$\delta_{ij} = \begin{cases} 1, & i = j \\ 0, & i \neq j \end{cases}$$

CHAP: IV

EQUATION OF CONTINUITY AND STRAIN ENERGY FUNCTION :

Equation of continuity in Eulerian form or equation of conservative of mass (In Eulerian form): '93 '90

Let P be any point of the continuum



$u, v, w \to$ component of velocity at the point P.

$\rho \to$ density at this point

$V \to$ vol. within S

$l, m, n \to$ d-cs of the outward drawn normal to the surface S

so $lu + mv + nw$ is the normal velocity of the material at this point of S outwards.

Therefore the rate at which material is entering within the vol. bounded by S accross the bounding surface is

$$-\iint \rho(lu + mv + nw)\, dS.$$

the rate of which material is accumulating within the volume is $\iiint_V \frac{\partial \rho}{\partial t} d\tau$ where $d\tau$ is the elementary vol. $dx\,dy\,dz$.

From the principle of conservation of mass these two rates must be equal

so,
$$\iiint_V \frac{\partial \rho}{\partial t} d\tau = -\iint_S (l\rho u + m\rho v + n\rho w)\, dS$$

$$= -\iiint_V \left[\frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] d\tau$$

[by Gauss's theorem]

or,
$$\iiint_V \left[\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w)\right] d\tau = 0$$

this is true for any vol. V which containing the pt. P

in its interior. Making the dimension of the vol. tends to zero in a manner so as to enclose the point P always we arrive at the conclusion that the integrand must vanish at point P.

therefore we have,
$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \quad \ldots (1)$$

this is the eqn of continuity in Eulerian form.

since P is any arbitrary point of the continuum, so the eqn (1) holds for every point of continuum. this can be written in vector notation as

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho \vec{q}) = 0 \quad \text{where} \quad \vec{q} = u\vec{i} + v\vec{j} + w\vec{k}$$

we write eqn (1) as

$$\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}\right) + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) = 0$$

when the material is incompressible

$$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} = \frac{D\rho}{Dt} = 0$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + w\frac{\partial}{\partial z}$

so in this case eqn of continuity is

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \quad \text{or} \quad \operatorname{div}\vec{q} = 0 \quad \ldots (2)$$

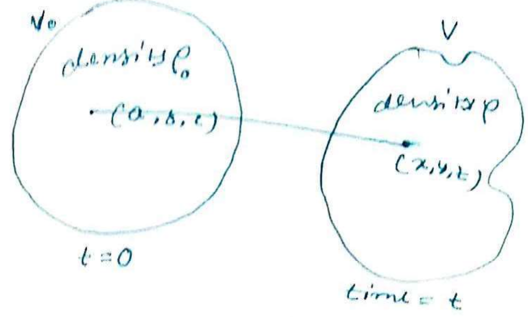
*** + (A-02)

Equation of continuity using Lagrange's system of Co-ordinates:

See → After Castigliano's formula

8.4.95

An alternative method of deriving eqn of continuity in Lagrangian form:



Let V be the volume at time t enclosing the same material particles which occupy the volume V_0 at time $t=0$. Let (x,y,z) be the co-ordinates of a material particle within volume V at time t which at $t=0$ was at (a,b,c) within V_0. Let ρ be the density of the material particle at (x,y,z) and ρ_0 at (a,b,c). Since the volume V and V_0 contained the same material particles so masses of the material within these two volumes must be equal. So,

$$\iiint_{V_0} \rho_0 \, da.db.dc = \iiint_{V} \rho \, dx.dy.dz \quad \ldots (1)$$

since (x,y,z) is the co-ordinate at time t of the particle which at $t=0$ was at (a,b,c). so (x,y,z) is a function of (a,b,c) if t be kept fixed. So changing the variable from (x,y,z) to (a,b,c) on R.H.S of (1)

$$\iiint_{V_0} \rho_0 \, da.db.dc = \iiint_{V_0} \rho . \frac{\partial(x,y,z)}{\partial(a,b,c)} da.db.dc$$

$$= \iiint_{V_0} \rho J \, da.db.dc$$

J is the Jacobian $\frac{\partial(x,y,z)}{\partial(a,b,c)}$

therefore, $$\iiint_{V_0} (\rho_0 - \rho J) \, da.db.dc = 0 .$$

since these relationship must hold for any V_0 ~~so that~~ so

it follows that $\rho_0 - \rho J = 0$

$$\Rightarrow \rho_0 = \rho J$$

this implies that ρJ is independent of time.

so. $\frac{d}{dt}(\rho J) = 0$

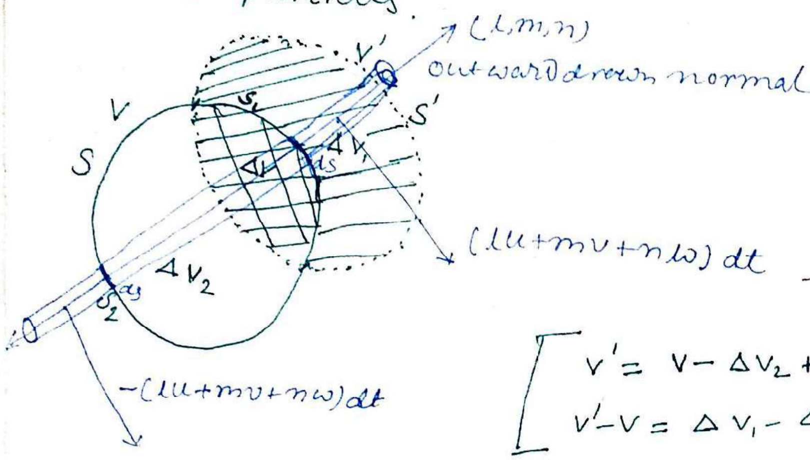
which is the eq^n of continuity in Lagrangian form.

<u>The material derivative of Volume Integral:</u>

Let $I(t)$ be the volume integral of a continuously differentiable function $A(x,y,z,t)$ which may be density, pressure, components of velocity or any physical quantity defined over a volume V occupied by a given set of particles at any time t. 294

$$\therefore \quad I(t) = \iiint A(x,y,z,t)\, dx\, dy\, dz$$

the function $I(t)$ is a function of time t because both the integrand $A(x,y,z,t)$ and V depend on the parameter t. the rate of change of $I(t)$ w.r.t. time denoted by $\frac{dI}{dt}$ or $\frac{DI}{Dt}$ is called the material derivative of $I(t)$, and is defined for a given set of material particles.



$$\left[\begin{array}{l} V' = V - \Delta V_2 + \Delta V_1 \\ V' - V = \Delta V_1 - \Delta V_2 = \Delta V \end{array}\right]$$

Let V be the volume enclosing the given set of material particles at time t and let S be its bounding surface (shown by solid line). At time $t+dt$ the material particles enclosed within V has moved to occupy the vol. V' and let the bounding surface of V' be S' (shown by doted line)

$$\therefore \quad \frac{DI(t)}{Dt} = \lim_{dt\to 0} \frac{I(t+dt) - I(t)}{dt}$$

$$= \lim_{dt\to 0} \frac{1}{dt}\left[\iiint_{V'} A(x,y,z,t+dt)\, dx\, dy\, dz - \iiint_{V} A(x,y,z,t)\, dx\, dy\, dz\right]$$

Let ΔV be the ~~domain~~ ~~V+V~~. so $\Delta V = \Delta V_1 - \Delta V_2$ where ΔV_1 is the newly occupied volume during the interval dt and ΔV_2 is the volume left from V during this interval.

Therefore,

$$\frac{DI}{Dt} = \lim_{dt\to 0} \frac{1}{dt}\left[\iiint_{V+\Delta V_1-\Delta V_2} A(x,y,z,t+dt)\,dv - \iiint_V A(x,y,z,t)\,dv\right]$$

$$= \lim_{dt\to 0} \left\{\frac{1}{dt}\left[\iiint_V \{A(x,y,z,t+dt) - A(x,y,z,t)\}\,dv + \iiint_{\Delta V_1} A(x,y,z,t+dt)\,dv - \iiint_{\Delta V_2} A(x,y,z,t+dt)\,dv\right]\right\}$$

$$= \iiint_V \frac{\partial A(x,y,z,t)}{\partial t} + \lim_{dt\to 0}\left\{\frac{1}{dt}\iiint_{\Delta V_1}\left[A(x,y,z,t) + \frac{\partial A(x,y,z,t)}{\partial t}dt + \cdots\right]dv\right\}$$

$$- \lim_{dt\to 0}\left\{\frac{1}{dt}\left[\iiint_{\Delta V_2} A(x,y,z,t) + \frac{\partial A(x,y,z,t)}{\partial t}dt + \cdots\right]dv\right\}$$

Since dt is infinitely small ΔV_1 and ΔV_2 are small so neglecting 2nd and higher order quantities we have

$$\frac{DI}{Dt} = \iiint_V \frac{\partial A(x,y,z,t)}{\partial t}\,dv + \lim_{dt\to 0}\frac{1}{dt}\iiint_{\Delta V_1} A(x,y,z,t)\,dv$$

$$- \lim_{dt\to 0}\frac{1}{dt}\iiint_{\Delta V_2} A(x,y,z,t)\,dv$$

For infinitesimal dt the last two integrals are approximated by taking the value of the integrand $A(x,y,z,t)$ on the surface S so that the integrals are equal to the sum of $A(x,y,z,t)$ multiplied by the volume swept by the particles situated on the boundary dS in time interval dt.

If (l,m,n) be the d-cs. of the outward drawn normal to the surface S at any point then $lu+mv+nw$ is the normal velocity at any point on S outwards. ~~Therefore~~

$$\therefore \iiint_{\Delta V_1} A(x,y,z,t)\,dv = \iint_{S_1} A(x,y,z,t)\,(lu+mv+nw)\,dt\,dS$$

$$\therefore \iiint_{\Delta V_1} A(x,y,z,t)\,dV = dt \iint_{S_1} [l(Au) + m(Av) + n(Aw)]\,ds$$

and
$$\iiint_{\Delta V_2} A(x,y,z,t)\,dV = \iint_{S_2} A(x,y,z,t).\{-(lu+mv+nw)\}\,dt\,ds$$

$$= -dt \iint_{S_2} [l(Au) + m(Av) + n(Aw)]\,ds$$

$$\therefore \frac{DI}{Dt} = \iiint_V \frac{\partial A(x,y,z,t)}{\partial t}\,dv + \iint_{S_1+S_2} [l(Au) + m(Av) + n(Aw)]\,ds$$

$$= \iiint_V \frac{\partial A(x,y,z,t)}{\partial t}\,dv + \iint_S [l(Au) + m(Av) + n(Aw)]\,ds$$

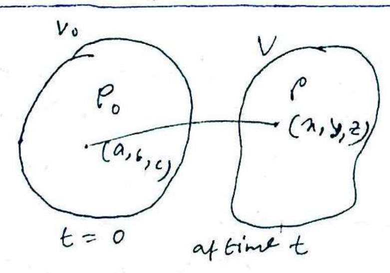
$$= \iiint_V \left[\frac{\partial A}{\partial t} + \frac{\partial (Au)}{\partial x} + \frac{\partial (Av)}{\partial y} + \frac{\partial (Aw)}{\partial z}\right] dV, \text{ by Gauss's theorem}$$

$$= \iiint_V \left[\left(\frac{\partial A}{\partial t} + u\frac{\partial A}{\partial x} + v\frac{\partial A}{\partial y} + w\frac{\partial A}{\partial z}\right) + A\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right]$$

$$= \iiint_V \left[\underset{\neq(P.Q)}{\frac{DA}{Dt}} + A\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] dv$$

So the material derivative is a volume integral containing the derivative of a physical quantity and a product of a physical quantity with divergence $\vec{q}$ i.e. div $\vec{q}$.

Derivation of equation of continuity using material derivative of volume Integral : 94



$V \rightarrow$ vol. at time t enclosing the same material particles which at $t=0$ occupied the vol. V_0.

Let (x,y,z) be the co-ordinates of a material particle at time t inside V, which at $t=0$ was at (a,b,c) in V_0. Let density at (x,y,z) be ρ whereas density at (a,b,c) is ρ_0.

So, $m_0 = \iiint_{V_0} \rho_0\, da\,db\,dc = \iiint_V \rho\, dx\,dy\,dz$

$\therefore$ ~~$\frac{d}{dt}$~~ $\frac{D}{Dt}\iiint_V \rho\, dx\,dy\,dz = 0$. [from the principle of conservation of mass] or, $\iiint_V \left[\frac{D\rho}{Dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right] dv = 0$

since this is true for any arbitrary volume V so the integrand must vanish.

$$\therefore \frac{D\rho}{Dt} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0$$

or, $$\frac{D\rho}{Dt} + \rho \frac{\partial v_i}{\partial x_j} = 0$$

or, $$\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z} + \rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) = 0$$

or, $$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

or, $\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \vec{q}) = 0$ where $\vec{q} = \vec{i}u + \vec{j}v + \vec{k}w$

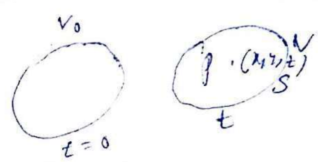
which is the eq^n of continuity in Eulerian form.

Equation of motion of a continuum applying the principles of Linear momentum:

294 991

$V_0 \to$ vol. at $t=0$ occupying a set of material particles

$V \to$ vol. at time t occupying the same set of material particles



$S \to$ bounding surface of V

$u, v, w \to$ compts. of velocity

$\rho \to$ density in V at time t at any point (x, y, z)

$\rho X, \rho Y, \rho Z \to$ compts. of body force $\rho \vec{F}$

$\tau_{\nu x}, \tau_{\nu y}, \tau_{\nu z} \to$ surface force (traction) on S at any point

$\vec{\nu} \to$ outward drawn normal to the surface with d-cs (l, m, n)

By Newton's 2nd Law, the rate of change of the component of linear momentum of the material within V in any direction must be equal to the component of the resultant of the forces on the material in V in that direction.

Considering components in x-direction

$$\frac{D}{Dt}\iiint u\rho\, dv = \iiint_V \rho X\, dv + \iint_S \tau_{\nu x}\, ds$$

or, $$\iiint_V \left[\frac{D}{Dt}(u\rho) + u\rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) \right] dv = \iiint_V \rho X\, dv + \iint_S (l\tau_{xx} + m\tau_{yx} + n\tau_{zx})\, ds$$

$$\iiint_V \left[\frac{D}{Dt}(u\rho) + u\rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right] dv = \iiint_V \rho X \, dv$$

$$+ \iiint_V \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) dv$$

[by Gauss's theorem]

$$\text{or, } \iiint_V \left[\frac{\partial}{\partial t}(u\rho) + u\frac{\partial}{\partial x}(u\rho) + v\frac{\partial}{\partial y}(u\rho) + w\frac{\partial}{\partial z}(u\rho)\right] + u\rho\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)$$

$$= \iiint_V \rho X \, dv + \iiint_V \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) dv$$

$$\text{or, } \iiint_V \Big[\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) + u\left(\frac{\partial \rho}{\partial t} + u\frac{\partial \rho}{\partial x} + v\frac{\partial \rho}{\partial y} + w\frac{\partial \rho}{\partial z}\right)$$

$$+ \rho u\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\Big] dv = \iiint_V \rho X \, dv + \iiint_V \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) dv$$

$$\text{or } \cancel{\iiint_V \left[\rho\frac{Du}{Dt} + u\left(\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right)\right.}$$

$$\text{or } \iiint_V \left[\rho\frac{Du}{Dt} + u\left\{\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z}\right\}\right] dv$$

$$= \iiint_V \rho X \, dv + \iiint_V \left(\frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z}\right) dv$$

$$\text{or, } \iiint \left[\rho\frac{Du}{Dt} - \rho X - \left(\frac{\partial}{\partial x}\tau_{xx} + \frac{\partial}{\partial y}\tau_{yx} + \frac{\partial}{\partial z}\tau_{zx}\right)\right] dv = 0$$

[since the quantity within the second bracket is equal to zero by the equation of continuity.]

Since the equation must hold for any arbitrary volume V so the integrand must vanish.

$$\therefore \rho\frac{Du}{Dt} = \frac{\partial}{\partial x}\tau_{xx} + \frac{\partial}{\partial y}\tau_{yx} + \frac{\partial}{\partial z}\tau_{zx} + \rho X \quad \ldots \; 3(a)$$

Similarly, considering the components of momentum and force in y and z directions respectively, we have

$$\rho\frac{Dv}{Dt} = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial}{\partial y}\tau_{yy} + \frac{\partial}{\partial z}\tau_{zy} + \rho Y \quad \ldots \; 3(b)$$

$$\rho\frac{Dw}{Dt} = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial}{\partial z}\tau_{yz} + \frac{\partial}{\partial z}\tau_{zz} + \rho Z \quad \ldots \; 3(c)$$

The eqns 3(a), 3(b) and 3(c) ~~are~~ together are called the

Eulerian eqⁿˢ ... continuum or the stress eqⁿ of motion of a continuum.

writing v_1, v_2, v_3 for u, v, w

X_1, X_2, X_3 for X, Y, Z

x_1, x_2, x_3 for x, y, z

$\tau_{11}, \tau_{22}, \tau_{12}$ etc. for $\tau_{xx}, \tau_{yy}, \tau_{xy}$ etc.

The eqns 3(a), 3(b) and 3(c) can be written as

$$\rho \frac{Dv_i}{Dt} = \frac{\partial \tau_{ji}}{\partial x_j} + \rho X_i \quad \ldots \quad (4) \quad i, j = 1, 2, 3$$

NOTE: a/ In case of viscous fluid the stress compts. τ_{xx}, τ_{yx} etc. are to be replaced by

$$\tau_{xx} = -p + \lambda\dot{\theta} + 2\mu \frac{\partial u}{\partial x}$$

$$\tau_{yy} = -p + \lambda\dot{\theta} + 2\mu \frac{\partial v}{\partial y}$$

$$\tau_{zz} = -p + \lambda\dot{\theta} + 2\mu \frac{\partial w}{\partial z}$$

$$\tau_{xy} = \mu\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right)$$

$$\tau_{yz} = \mu\left(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}\right)$$

$$\tau_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)$$

where λ and μ are the viscosity co-efficient of the fluid.

b/ In case of non-viscous incompressible fluid i.e. perfect fluid; $\mu = 0$, $\dot{\theta} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

so, $\tau_{xx} = -p$, $\tau_{yy} = -p$, $\tau_{zz} = -p$

and $\tau_{xy} = \tau_{yz} = \tau_{zx} = 0$

So the equations 3(a), 3(b), 3(c) become

$$\frac{Du}{Dt} = X - \frac{1}{\rho}\frac{\partial p}{\partial x}$$

$$\frac{Dv}{Dt} = Y - \frac{1}{\rho}\frac{\partial p}{\partial y}$$

$$\frac{Dw}{Dt} = Z - \frac{1}{\rho}\frac{\partial p}{\partial z}$$

these are called Euler's equations of motion for perfect fluid.

c/ In case of motion of elastic bodies, τ_{xx}, τ_{xy} etc. are to be replaced by

$$\tau_{xx} = \lambda\theta + 2\mu \frac{\partial u_x}{\partial x}$$

$$\tau_{yy} = \lambda\theta + 2\mu \frac{\partial u_y}{\partial y}$$

$$\tau_{zz} = \lambda\theta + 2\mu \frac{\partial u_z}{\partial z}$$

$$\tau_{xy} = \mu\left(\frac{\partial u_y}{\partial x} + \frac{\partial u_x}{\partial y}\right)$$

$$\tau_{yz} = \mu\left(\frac{\partial u_z}{\partial y} + \frac{\partial u_y}{\partial z}\right)$$

$$\tau_{zx} = \mu\left(\frac{\partial u_x}{\partial z} + \frac{\partial u_z}{\partial x}\right)$$

where u_x, u_y, u_z are the components of displacement and

$$\theta = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} \quad [\equiv e_{xx} + e_{yy} + e_{zz}].$$

In linear theory of elasticity we assume that the displacement components u_x, u_y, u_z and the velocity compts. u, v, w together with their derivatives are small.

$$\therefore u = \frac{Du_x}{Dt} = \frac{du_x}{dt} = \frac{\partial u_x}{\partial t} + u\frac{\partial u_x}{\partial x} + v\frac{\partial u_x}{\partial y} + w\frac{\partial u_x}{\partial z}$$

$$= \frac{\partial u_x}{\partial t}, \text{ neglecting 2nd order quantities}$$

similarly $v = \dfrac{\partial u_y}{\partial t}$ and $w = \dfrac{\partial u_z}{\partial t}$

and the acceleration compts. are $\cancel{\frac{Du}{Dt}}$ $\Rightarrow$

$$\frac{Du}{Dt} = \frac{du}{dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}$$

$$= \frac{\partial^2 u_x}{\partial t^2}$$

similarly, $\dfrac{Dv}{Dt} = \dfrac{\partial^2 u_y}{\partial t^2}$ and $\dfrac{Dw}{Dt} = \dfrac{\partial^2 u_z}{\partial t^2}$

Thus in elasto-dynamics, the equations of motion are

$$\rho\frac{\partial^2 u_x}{\partial t^2} = \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{yx}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + \rho X \qquad \text{9 Q4}$$

$$\rho\frac{\partial^2 u_y}{\partial t^2} = \frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{zy}}{\partial z} + \rho Y$$

$$\rho\frac{\partial^2 u_z}{\partial t^2} = \frac{\partial \tau_{xz}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z} + \rho Z$$

or, $\rho\dfrac{Dv_i}{Dt} = \dfrac{\partial \tau_{ji}}{\partial x_j} + \rho X_i$, In Elasto-statics $\dfrac{\partial \tau_{ji}}{\partial x_j} + \rho X_i = 0$

These are the eq's of equilibrium used in solid mechanics.

principle of Conservation of Energy: 87, 89, 93

$V_0 \to$ Vol. at $t=0$ occupying a set of material particles.

$V \to$ Vol. at time t occupying same set of material particles.

$v_1, v_2, v_3 \to$ compts. of velocity.

$\rho \to$ density at any pt. (x_1, x_2, x_3) in V.

$\rho X_1, \rho X_2, \rho X_3 \to$ compts. of body force $\rho\vec{F}$.

$\tau_{\nu 1}, \tau_{\nu 2}, \tau_{\nu 3} \to$ compts. of surface force at any pt. on S.

$\vec{\nu} \to$ outward drawn normal with d-cs (l, m, n).

* The principle of conservation of energy states that the time rate of change of kinetic and internal energy of the material within V must be equal to the rate of work done by the body and surface forces plus any non-mechanical energy supplied to the material within V per unit time

Non-mechanical energy may include thermal, chemical, electro-magnetic energy

we shall consider here only thermal energy change.

Let K and E be the kinetic and internal energy respectively.

The energy principle gives

$$\frac{dK}{dt} + \frac{dE}{dt} = \text{rate of work done by body and surface forces on the material in V + rate of increase of total heat within the material in V.} \quad \cdots (1)$$

Now $$\frac{dK}{dt} = \frac{d}{dt}\iiint_V \frac{1}{2}\rho q^2 \, dv, \quad q^2 = v_1^2 + v_2^2 + v_3^2$$ and ρq^2 is a physical quantity.

$$= \iiint_V \left[\frac{D}{Dt}\left(\frac{1}{2}\rho q^2\right) + \frac{1}{2}\rho q^2\left(\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3}\right)\right] dv$$

$$= \iiint_V \left[\frac{1}{2}q^2\left\{\frac{D\rho}{Dt} + \rho\frac{\partial v_i}{\partial x_i}\right\} + \frac{\rho}{2}\frac{D}{Dt}(v_1^2 + v_2^2 + v_3^2)\right] dv$$

$$= \iiint \rho\left[v_1\frac{Dv_1}{Dt} + v_2\frac{Dv_2}{Dt} + v_3\frac{Dv_3}{Dt}\right] dv.$$

since by the eq^n of continuity $\frac{D\rho}{Dt} + \rho\frac{\partial v_i}{\partial x_i} = 0$

So, $$\frac{dK}{dt} = \iiint_V \rho\left[v_j\frac{Dv_j}{Dt}\right] dv \quad \cdots\cdots (2)$$

Let $e =$ internal energy per unit mass